Lecture 26

11.6 - Absolute Convergence and the Root and Ratio Tests

Absolute Convergence

A series $\sum a_n$ is called <u>absolutely convergent</u> if the series $\sum |a_n|$ converges.

<u>Ex</u>: Are the series absolutely convergent?

ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)+(-1)^{n-1}}{n^2}$ ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

ⓐ $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos(n)+(-1)^{n-1}}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{2}{n^2}$ So $\sum |a_n|$ converges by the comparison test $\Rightarrow \sum a_n$ conv. abs.

ⓑ $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by the p-series test.

So, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ does not converge absolutely.

<u>Def</u>: A series which is convergent, but not absolutely convergent is called <u>conditionally convergent</u>.

Theorem: If $\sum a_n$ is absolutely convergent, then it is convergent.

proof: $\sum a_n$ absolutely convergent $\Rightarrow \sum |a_n| < \infty$. Let's say $\sum |a_n| = S$. Since $-|a_n| \leq a_n \leq |a_n|$ for all n:

$$-\sum |a_n| = \sum (-|a_n|) \leq \sum a_n \leq \sum |a_n| \Rightarrow -S \leq \sum a_n \leq S$$

So, $\sum a_n$ converges.

Ex: Do the series converge?

ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{17}}$ ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$

ⓐ $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^{17}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{17}}$, which converges by p-series test

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{17}}$ is (absolutely) convergent.

ⓑ $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^3}\right| \underset{(n\geq 1)}{=} \sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^3} \underset{(|\sin(n)|\leq 1)}{\leq} \sum_{n=1}^{\infty} \frac{1}{n^3}$, convergent by comparison test.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ is (absolutely) convergent

Ex: When is the series $\sum_{n=1}^{\infty} ar^n$ absolutely convergent?

$\sum_{n=1}^{\infty} |ar^n| = \sum_{n=1}^{\infty} |ar||r|^{n-1}$ geometric, so it converges (absolutely) if $|r|<1$. (In particular, if a geometric series converges, it does so absolutely.)

The Ratio Test: Consider the series $\sum_{n=1}^{\infty} a_n$, and let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

- If $L<1$, the series converges absolutely.
- If $L>1$, the series diverges
- If $L=1$, the test is inconclusive.

This test is useful when we have factorials and powers of constants hanging around.

Ex: Test the following series for convergence:

ⓐ $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ ⓑ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(1.1)^n}{n^4}$ ⓒ $\displaystyle\sum_{n=6}^{\infty} \frac{1}{n^3}$ ⓓ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

using the ratio test.

Sol:

ⓐ $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-3)^n}\right| = \lim_{n\to\infty} \frac{3}{(2n+2)(2n+3)} = 0 < 1$$

absolutely convergent

ⓑ $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+2}(1.1)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(-1)^{n+1}(1.1)^n}\right| = \lim_{n\to\infty} \frac{1.1\, n^4}{(n+1)^4} = 1.1 > 1$$

divergent

ⓒ $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1$$ inconclusive

(but we know this converges by the p-series test)

ⓓ $$\lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$ inconclusive.

(but we know this diverges by the p-series test)

The Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$ and let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

- If $L<1$, the series converges absolutely.
- If $L>1$, the series diverges.
- If $L=1$, the test is inconclusive.

Ex: Use the root test on the series:

ⓐ $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ ⓑ $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ ⓒ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

Sol:

ⓐ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$ absolutely convergent

ⓑ $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^{5n}} = \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^5 = 2^5 > 1$ divergent.

ⓒ $\lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{n^2}\right|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$, divergent.

Ex: In section 11.10, we will learn how to show that

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

For what values of x is this a valid equation? (I.e., the series is convergent.) Does the convergence type (absolute/conditional) depend on x?

Sol: Using the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^n x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} x^n} \right| = \lim_{n\to\infty} \frac{|x|\, n}{n+1} = |x|$$

So, the series converges absolutely if $|x|<1$.

What if $|x|=1$?

$x=1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, a convergent alternating series

$x=-1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ divergent by p-series test.

So, the equation is valid for $-1 < x \leq 1$.

Rearrangements

The convergence of a series which is conditionally convergent is quite delicate... The last example shows

$$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

but, if we arrange the terms as:

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \cdots$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots\right]$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{2} \ln 2$$

Interesting Facts

- If $\sum a_n$ is abs. conv. & $\sum a_n = S$, then any rearrangement of terms still sums to S.
- If $\sum a_n$ is cond. conv., then for any real number r, we can find a rearrangement so that the sum is r.